Group Representation of Global Intrinsic Symmetries

Hui Wang\textsuperscript{1,2} and Hui Huang\textsuperscript{1,2}

\textsuperscript{1} School of Information Science and Technology, Shijiazhuang Tiedao University, China
\textsuperscript{2} College of Computer Science & Software Engineering, Shenzhen University, China

Abstract

Global intrinsic symmetry detection of 3D shapes has received considerable attentions in recent years. However, unlike extrinsic symmetry that can be represented compactly as a combination of an orthogonal matrix and a translation vector, representing the global intrinsic symmetry itself is still challenging. Most previous works based on point-to-point representations of global intrinsic symmetries can only find reflectional symmetries, and are inadequate for describing the structure of a global intrinsic symmetry group. In this paper, we propose a novel group representation of global intrinsic symmetries, which describes each global intrinsic symmetry as a linear transformation of functional space on shapes. If the eigenfunctions of the Laplace-Beltrami operator on shapes are chosen as the basis of functional space, the group representation has a block diagonal structure. We thus prove that the group representation of each symmetry can be uniquely determined from a small number of symmetric pairs of points under certain conditions, where the number of pairs is equal to the maximum multiplicity of eigenvalues of the Laplace-Beltrami operator. Based on solid theoretical analysis, we propose an efficient global intrinsic symmetry detection method, which is the first one able to detect all reflectional and rotational global intrinsic symmetries with a clear group structure description. Experimental results demonstrate the effectiveness of our approach.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computer Graphics/Computational Geometry and Object Modeling—[Geometric algorithms, languages, and systems]

1. Introduction

Symmetries are universal phenomena, in both natural and man-made shapes, which reflect high-level information about shape structure. Many applications in geometric modeling and processing utilize the symmetry information, such as, segmentation [SKS06], feature correspondence [LKF12], shape matching [KFR04, THW\textsuperscript{*}14], function matching [AVBC13], geometry completion [TW05] and meshing [PGR07]. The characterization and detection of symmetries of 3D shapes, thus receives significant attention in computational geometry and computer graphics; a comprehensive review is provided in [MPWC13].

\textsuperscript{†} Corresponding author: Hui Huang (hhzhiiyan@gmail.com)
The symmetry of a shape can be regarded as a distance preserving self-homeomorphism. Based on types of distances, symmetry can be classified into extrinsic [MGP06, PSG10, MSHS06, LJYL16, SAD16] and intrinsic [RBBK07, OSG08, XZT09, KLC10, XZJ12]. While extrinsic symmetry detection finds rigid transformations that are Euclidean distance preserving, intrinsic symmetry detection uses geodesic distances and looks for isometric deformations. Symmetry can also be categorized as global [MSHS06, OSG08] and partial [XZT09, XZJ12] by taking consideration all or part of the shape. In this paper, we focus on the characterization and detection of global intrinsic symmetries on 3D shapes, which can be represented as compact manifolds.

While extrinsic symmetry can be compactly represented using an orthogonal matrix and a translation vector, i.e., just six degrees of freedom, detecting and representing intrinsic symmetry is much more challenging, in particular for rotational intrinsic symmetries. The most frequently used symmetry representation is the point-to-point correspondence, which indicates each point and its symmetric point as a pair. However, it is generally intractable to determine symmetric pairs of points under mild conditions shown in Figure 2, where the minimal number of pairs is the maximum multiplicity of the eigenvalue of the Laplace-Beltrami operator.

Based on the eigen-decomposition of the Laplace-Beltrami operator, Ovsjanikov et al. [OSG08] transform intrinsic symmetries on the shape into extrinsic symmetries in a high dimensional signature space. The intrinsic symmetry can be uniquely determined from a single pair of points using the proposed Heat Kernel Map [OMMG10]. However, the non-repeated eigenvalue restriction of the Laplace-Beltrami operator causes these two proposed algorithms to fail for shapes with rotational intrinsic symmetry order larger than two, due to repeated eigenvalues [Ovs11].

Kim et al. [KLC10] use low dimensional Möbius transformations to find intrinsic symmetries. However, their method is limited on shapes with zero genus and also relies on symmetry representation using point-to-point correspondences. Based on the state-of-the-art functional map [OBCS12], Liu et al. [LLL15] propose an intrinsic symmetry detection method that still fails to detect rotational intrinsic symmetries.

In this paper, inspired by the classical group representation theory [TGT07, Gur08] and the functional map in geometry processing [OBCS12], we represent each global intrinsic symmetry as a linear transformation on the function space defined on the shape (see Figure 1), and have proven that each global intrinsic symmetry can be uniquely determined from sparse symmetric pairs of points under certain conditions. Based on these analyses, we propose a novel algorithm for global intrinsic symmetry detection.

Our main technical contributions can be summarized as:

- Represent each global intrinsic symmetry of a compact manifold in any dimension as a linear transformation on the function space defined on the manifold, which can be represented by a matrix under a chosen basis on the function space. This allows us to study the structure of the global intrinsic symmetry group through the use of simple linear algebra, e.g., the composition shown in Figure 2.
- Prove that, if the eigenfunctions of the Laplace-Beltrami operator are chosen as the basis for the function space, each global intrinsic symmetry can be uniquely recovered from some sparse symmetric pairs of points under mild conditions shown in Figure 3, where the minimal number of pairs is the maximum multiplicity of the eigenvalue of the Laplace-Beltrami operator. This provides a new band-to-band method to compute the functional map using much fewer constraints than previous works.

Figure 2: The composition operation of global intrinsic symmetries is equal to a product of their $21 \times 21$ representation matrices. The middle row presents the symmetries via transferring colors from the top one onto the shape itself. Each matrix at bottom is the functional representation of the symmetry, as shown by the above.

Figure 3: Both reflectional (top) and rotational (bottom) global intrinsic symmetries can be uniquely determined from a small number of symmetric pairs of points.
• Propose a fast symmetry detection algorithm that can not only handle reflectional symmetries but also rotational global intrinsic symmetries as demonstrated in Figure 1.

2. Related work

The literature on symmetry representation and detection is vast [MPWC13]. Here we mainly concentrate on representation of global intrinsic symmetries of 3D shapes and briefly introduce existing methods of detecting them.

Point-to-point correspondence. Global intrinsic symmetry is an isometric mapping on a shape, which preserves geodesic distances between every two points. Perhaps the most common representation of global intrinsic symmetry is the point-to-point correspondence, i.e., representing each point and its symmetric point as a pair. It is difficult to investigate the symmetry group structure, such as computing composition and order of symmetries, using this representation in practice. In contrast, our proposed representation allows much simpler linear algebra computation to delve into the structure of the symmetry group, such as composition, group table, inverse and order of each symmetry, etc. Furthermore, searching symmetries directly based on the point-to-point correspondences is an NP-hard subclass of the quadratic assignment problem. The search space can be greatly reduced by ruling out bad mappings that cannot preserve local geometric signatures and global consistent distances [RBBK07, RBBK10, RBB+10]. However, these methods are still quite computationally expensive and it is difficult to handle shapes with multiple symmetries [MPWC13].

Möbius transformation. Based on the observation that the group of global intrinsic symmetry is a subset of anti-Möbius transformations, which has only six degrees of freedom for genus zero shapes, Kim et al. [KLC10] and Panuzzo et al. [PLPZ12] propose novel intrinsic symmetry detection methods. Among candidate Möbius transformations the one that best maps the shape onto itself is selected as the detected symmetry. Those candidate Möbius transformations are generated by enumerating small subsets of a symmetry invariant point set, which are critical points of average geodesic distance functions. The candidate maps can be blended to a better one [KLF11], leading to a higher dimension matrix and thus are computationally costly. These methods are limited to genus zero shapes, whereas our approach is much faster and applicable to shapes of any genus and dimension under light conditions. Furthermore, these methods rely on representing symmetries as point-to-point correspondences, which restrict their ability of investigating the structure of a symmetry group.

Global point signature. Ovsjanikov et al. [OSG08] prove that global intrinsic symmetries on a compact manifold can be converted to Euclidean symmetries on global point signatures via the eigen-decomposition of the Laplace-Beltrami operator [Rus07]. Through this theory, they propose a symmetry detection algorithm on a restricted signature space, where each symmetry is represented as a sequence of signs of non-repeating eigenfunctions. However, if a compact manifold has a rotational symmetry g such that $g^2 \neq 1$, the eigenvalues of the Laplace-Beltrami operator of the manifold must contain some repetition [Ovs11]. Therefore, this method fails to detect rotational global intrinsic symmetries with an order larger than two.

Heat kernel map. Ovsjanikov et al. [OMMG10] point out that global intrinsic symmetry can be uniquely recovered from only one symmetric pair of points for shapes that only have non-repeated eigenvalues of the Laplace-Beltrami operator. This argument is consistent with our proposed group representation. That is, the group representation of a global intrinsic symmetry can be uniquely determined from one symmetric pair of points for shapes without repeated eigenvalues. However, this restriction on non-repeated eigenvalues limits the ability of this method to detect rotational symmetries, such as the one in [OSG08], where the symmetries are still represented as point-to-point correspondences.

Functional map. The functional map [OBCS+12] has been widely used in geometry processing [OCB+16], such as shape correspondence [PBB+13], map visualization [OBCCG13], symmetry detection [LLL+15], measuring shape difference [ROA+13, CSBC+17], etc. In particular, the global intrinsic symmetries on one reference shape can be transferred into another target shape based on decomposing the functional map into two parts, which act respectively on the space of symmetric functions and its orthogonal complement [OMPG13]. However, the symmetries on the reference shape have to be known in advance.

Note that almost all previous works need to add many constraints to compute the functional map entirely, e.g., sparsity [PBB+13], orthogonality [LLL+15], commutativity with specific operators [NO17]. Nonetheless, they lack the theoretical analysis to guarantee at least how many constraints are enough to uniquely determine the functional map. Our proposed method is built from the functional map. Meanwhile we have proven that the functional map of a global intrinsic symmetry can be uniquely recovered from a small number of symmetric pairs of points under certain mild conditions. This benefits from when we decompose the functional map into the direct sum of several parts, where each part acts respectively on the eigenfunction space of each eigenvalue (repeated or non-repeated) and thus can be computed separately with much fewer constraints.

Group representation. Analyzing a symmetry group through the induced linear transformation on the function space is a classical approach in the Representation Theory [TGT07], where group elements can be represented as matrices so that the group operation can be represented by a matrix product. Many group-theoretic problems can, therefore, be reduced into much simpler problems in linear algebra. The theory about relations of symmetry groups, representations and Laplacians can also refer to [Gur08]. Inspired by these analyses, we use the representation theory to analyze and detect the global intrinsic symmetries on manifolds.

3. Theoretical analysis

In this section, based on the classical representation theories [TGT07, Gur08], we analyze group representation of global intrinsic symmetries on a continuous compact Riemannian manifold in any dimension and its properties when choosing eigenfunctions of the Laplace-Beltrami operator as the basis of the function.
space defined on the manifold. For a compact Riemannian manifold \( M \) with the standard measure induced by the volume form, the space of square-integrable functions on the manifold \( M \) is denoted by \( L^2(M) = \{ f : M \rightarrow \mathbb{R} \mid \int_M f^2 ds < \infty \} \) with the inner product \(< f, g >_M = \int_M f g ds\). All of the bijective linear transformations \( L^2(M) \rightarrow L^2(M) \) on the function space form a group, written as \( GL(L^2(M)) \), with composition as a group operation.

### 3.1. Global intrinsic symmetry group

A self-homeomorphism \( g : M \rightarrow M \) on the compact Riemannian manifold \( M \) that preserves geodesic distances is called a global intrinsic symmetry

\[
d(p, q) = d(g(p), g(q)), \forall p, q \in M, \tag{1}
\]

where \( d(p, q) \) is the geodesic distance of points \( p \) and \( q \) [RBBK07].

The set of all global intrinsic symmetries of the manifold \( M \) forms a group \( G(M) \) with composition as the group operation. Obviously, the identity mapping \( I \) on \( M \) is an element of \( G(M) \).

### 3.2. Functional representation of global intrinsic symmetry

Inspired by the functional map for shape correspondences [OBCS’12], each global intrinsic symmetry \( g \in G(M) \) on the manifold \( M \) induces a bijective linear transformation \( F^g \in GL(L^2(M)) \) on the space of square-integrable functions \( L^2(M) \) as

\[
F^g(f)(p) = f(g(p)), \tag{2}
\]

for any function \( f \in L^2(M) \) and point \( p \in M \). The above map \( F^g \) is usually called the functional representation or functional map of the symmetry \( g \). We list the following three propositions that are directly taken from the original functional map framework [OBCS’12]. To make the paper self-contained, we provide here the proof of the proposition 3.3.

**Proposition 3.1.** The original symmetry \( g \) can be recovered from its functional representation \( F^g \).

**Proposition 3.2.** For every symmetry \( g \in G(M) \), \( F^g \) is a linear transformation on the function space \( L^2(M) \).

**Proposition 3.3.** Given a basis \( \phi_1, \phi_2, \ldots, \phi_n \) on the function space \( L^2(M) \), for each symmetry \( g \in G(M) \), \( F^g \) can be represented as an infinite matrix \( C^g \) with element \( C^g_{ji} = \langle F^g(\phi_j), \phi_i \rangle \) s.t. the coefficients vector of \( F^g(f) \) is \( C^g a \), where \( f \) is a function with coefficients vector \( a \).

**Proof** Suppose that \( L^2(M) \) is equipped with a basis \( \phi_1, \phi_2, \ldots, \phi_n \), so that \( f \in L^2(M) \) can be represented as a linear combination of the basis \( f = \sum a_i \phi_i \). Then,

\[
F^g(f) = F^g(\sum a_i \phi_i) = \sum a_i F^g(\phi_i). \tag{3}
\]

\( F^g(\phi_i) \) is also a function on the manifold \( M \) and can be represented as \( F^g(\phi_i) = \sum C^g_{ji} \phi_j \), where \( C^g_{ji} = \langle F^g(\phi_j), \phi_i \rangle \). Therefore,

\[
F^g(f) = \sum a_i \left( \sum C^g_{ji} \phi_j \right) = \sum \sum a_i C^g_{ji} a_j \phi_j. \tag{4}
\]

The vectors of coefficients of functions \( f \) and \( F^g(f) \) are \( a = (a_1, a_2, \ldots, a_i, \ldots)^T \) and \( C^g a \) respectively. \( \square \)

### 3.3. Representation of global intrinsic symmetry group

Based on the propositions in the previous section, each global intrinsic symmetry \( g \in G(M) \) induces a bijective linear transformation \( F^g \in GL(L^2(M)) \). We can then define a map \( \rho : G(M) \rightarrow GL(L^2(M)) \) as follows

\[
\rho(g) = F^g, \forall g \in G(M). \tag{5}
\]

**Proposition 3.4.** The map \( \rho : G(M) \rightarrow GL(L^2(M)) \) is a faithful group representation of \( G(M) \).

**Proof** \( \forall g_1, g_2 \in G(M) \), \( \forall f \in L^2(M) \) and \( p \in M \), we have \( \rho(g_1g_2)(f)(p) = F^{g_2}(f)(g_1(p)) = F^{g_1}(f)(g_2(p)) \). Therefore, \( \rho(g_1g_2) = \rho(g_1) \circ \rho(g_2) \), i.e., \( \rho(g_1g_2) = \rho(g_1) \circ \rho(g_2) \). Therefore, the map \( \rho : G(M) \rightarrow GL(L^2(M)) \) is a representation of symmetry group \( G(M) \).

**Proposition 3.5.** Given a basis \( \phi_1, \phi_2, \ldots, \phi_n \) on \( L^2(M) \), \( G^*(M) = \{ C^*, \forall g \in G(M) \} \) is a matrix group, which is an isomorphism of the symmetry group \( G(M) \).

**Proof** For the basis \( \phi_1, \phi_2, \ldots, \phi_n \) on \( L^2(M) \), \( F^g \in GL(L^2(M)) \) can be represented as an infinite matrix \( C^g \) defined in Equation (4). Furthermore, since \( \rho : G(M) \rightarrow GL(L^2(M)) \) is a faithful group representation, each global intrinsic symmetry \( g \in G(M) \) can be represented as an invertible matrix \( C^g \) and \( G(M) \) is an isomorphism of the matrix group \( G^*(M) = \{ C^g, \forall g \in G(M) \} \).

The above group representation of \( G(M) \) allows that many group-theoretic problems can be reduced to the much simpler problems in linear algebra. Instances include:

- Composition operation on group \( G(M) \) can be represented by the matrix product on \( G^*(M) \), that is, \( \forall g_1, g_2 \in G(M) \), the matrix representation of \( g_1, g_2 \) is \( C^g_1 \cdot C^g_2 \) shown in Figure 2.
- The matrix representation of identity symmetry \( I \) of the manifold \( M \) is the identity matrix \( I \).
- For all symmetry \( g \in G(M) \), the matrix representation of its inverse element \( g^{-1} \) is \( C^g^{-1} \).

If symmetry \( g \in G(M) \) has finite order \( m \), then its representation matrix \( C^g \) also has the same order \( m \), and vice versa.

### 3.4. Basis of the Laplace-Beltrami eigenfunctions

Suppose the Laplace-Beltrami operator \( \Delta_M \) on the compact Riemannian manifold \( M \) has eigenvalues \( 0 = \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \), where \( \lambda_n \) corresponds to an \( n \)-dimensional eigenfunction space \( W_n \) with an orthogonal basis defined as \( \phi_1, \ldots, \phi_n \). Then it is known that \( \phi_1, \phi_1', \phi_1'', \ldots, \phi_2, \ldots, \phi_n, \ldots \) forms an orthogonal basis of \( L^2(M) \). Beside the compactness and stability characteristics in the original functional map framework [OBCS’12], the group representations of global intrinsic symmetries have some other special properties under the above orthogonal basis.

© 2017 The Author(s)

Computer Graphics Forum © 2017 The Eurographics Association and John Wiley & Sons Ltd.
Proposition 3.6. For each symmetry \( g \in G(M) \), the eigenfunction space \( W_i \) is an invariant space under the linear transformation \( F^g \). Furthermore, its representation matrix \( C^g \) is orthogonal, i.e., \( C^g(C^g)^T = (C^g)^T C^g = I \), and has a block diagonal structure, where the \( i \)-th block matrix \( D_i^g \) is with dimension \( i_k \).

Proof: For the orthogonal basis \( \phi_1, \phi_2, \ldots, \phi_{i_k} \) of the space \( W_i \), \( F^g(\phi_1), F^g(\phi_2), \ldots, F^g(\phi_{i_k}) \) is also an orthogonal basis of \( W_i \) [Ros97, OSG08]. Thus, there exists an \( i_k \)-dimensional orthogonal matrix \( D_i^g \) such that

\[
(F^g(\phi_1), F^g(\phi_2), \ldots, F^g(\phi_{i_k})) = (\phi_1, \phi_2, \ldots, \phi_{i_k}) D_i^g.
\]

Then \( C^g \) is also an orthogonal matrix with a block diagonal structure, whose \( i \)-th block matrix is \( D_i^g \). \( \square \)

The above proposition means \( L^2(M) \) can be split into the direct sum of the eigenfunction spaces \( L^2_i(M) = \bigoplus W_i \), which are invariant under all linear transformations \( F^g \) induced by \( g \in G(M) \). That is, \( F^g \) can be decomposed into the following direct sum

\[
F^g = \bigoplus_i F_i^g, \tag{6}
\]

where \( F_i^g \) is the induced linear transformation of \( F^g \) on the functional space \( W_i \), and the representation matrix of \( F_i^g \) is \( D_i^g \). If \( f \in L^2(M) \) and its projection on the subspace \( W_i \) is \( \sum_{j=1}^{i_k} a_j \phi_j \) with a coefficient vector \( a_i = (a_1, a_2, \ldots, a_{i_k})^T \), then the coefficient vector \( b_i \) of function \( F_i^g(\sum_{j=1}^{i_k} a_j \phi_j) \)

\[
b_i = D_i^g a_i, \tag{7}
\]

If we define the coefficient vector of indicator function \( f_p \) of point \( p \in M \) as its spectral embedding

\[
\Phi(p) = (\phi_1(p), \phi_2(p), \ldots, \phi_{i_k}(p))^T, \tag{8}
\]

then for \( g \in G(M) \) with its representation matrix \( C^g \), we get \( \Phi(g(p)) = C^g \Phi(p) \) and

\[
(\phi_1(g(p)), \ldots, \phi_{i_k}(g(p))) = D_i^g(\phi_1(p), \ldots, \phi_{i_k}(p))^T. \tag{9}
\]

Each global intrinsic symmetry \( g \in G(M) \) induces an extrinsic symmetry on the above spectral embedding, which can be represented as the orthogonal matrix \( C^g \) with a block diagonal structure.

3.5. Main theorem

Based on the above, if we choose eigenfunctions of the Laplace-Beltrami operator for the basis of the function space \( L^2(M) \) on the manifold \( M \), then the functional representation \( F^g \) of every global intrinsic symmetry \( g \in G(M) \) can be decomposed into the direct sum of some linear transformation as in Equation (6). Unlike previous works solving the representation matrix \( C^g \) of functional map \( F^g \) entirely with many constraints, we compute its block matrix \( D_i^g \) corresponding to the induced functional map \( F_i^g \) separately. This strategy has the advantage that each matrix \( D_i^g \) can be computed with a much smaller number of constraints. In this section, we prove that the global intrinsic symmetry can be uniquely determined by only a small number of constraints.

Definition 3.1. Functions \( f_j, j = 1, 2, \ldots, J \) are called full rank if the rank of their projections on the eigenfunction space \( W_i \) of the eigenvalue \( \lambda_i \) of the Laplace-Beltrami operator is \( i_k, j = 1, 2, \ldots \).

It is obvious that functions \( f_j, j = 1, 2, \ldots, J \) are full rank if and only if \( \text{rank}(E_i) = i_k \), the coefficient matrix \( E_i \) is defined as

\[
E_i = \begin{pmatrix}
a_{11}^i & a_{12}^i & \cdots & a_{1i_k}^i \\
a_{21}^i & a_{22}^i & \cdots & a_{2i_k}^i \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_k1}^i & a_{i_k2}^i & \cdots & a_{i_ki_k}^i
\end{pmatrix} , \tag{10}
\]

where \( a_{ij}^i \phi_1 + a_{ij}^i \phi_1 + \ldots + a_{ij}^i \phi_{i_k} \) is the projection of function \( f_j \) on the space \( W_i, j = 1, 2, \ldots, J \).

Theorem 3.1. If the eigenvalues of the Laplace-Beltrami operator on the compact manifold \( M \) with a bounded maximum multiplicity \( m, i.e., m = \max\{i_1, i_2, \ldots, i_{i_k}\}, \) then for each global intrinsic symmetry \( g \in G(M) \), \( m \) full rank functions \( f_j \) and their mapping functions \( F^g(f_j), j = 1, 2, \ldots, J \) under the action of \( g \) can uniquely determine the symmetry representation matrix \( C^g \).

Proof: Suppose \( a_{11}^i \phi_1 + a_{12}^i \phi_1 + \ldots + a_{1i_k}^i \phi_{i_k} \) is the projection of function \( f_j \) on the space \( W_i \), and \( F^g(f_j) \) is represented as \( F^g(f_j) = b_{11}^i \phi_1 + b_{12}^i \phi_1 + \ldots + b_{i_k}^i \phi_{i_k}, j = 1, 2, \ldots, m \).

Based on Equation (7), we have

\[
\begin{pmatrix}
b_{11}^i & b_{12}^i & \cdots & b_{1m}^i \\
b_{21}^i & b_{22}^i & \cdots & b_{2m}^i \\
\vdots & \vdots & \ddots & \vdots \\
b_{i_k1}^i & b_{i_k2}^i & \cdots & b_{i_km}^i
\end{pmatrix} = D_i^g \begin{pmatrix}
a_{11}^i & a_{12}^i & \cdots & a_{1i_k}^i \\
a_{21}^i & a_{22}^i & \cdots & a_{2i_k}^i \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_k1}^i & a_{i_k2}^i & \cdots & a_{i_ki_k}^i
\end{pmatrix} . \tag{11}
\]

Since \( m \geq i_k \) and functions \( f_j, j = 1, 2, \ldots, m \) are full rank, there exists \( i_k \) linear independent columns from the right matrix \( J \). If the column indexes are \( j_1, j_2, \ldots, j_{i_k} \), the following matrix is invertible

\[
\begin{pmatrix}
a_{j_11}^i & a_{j_12}^i & \cdots & a_{j_1i_k}^i \\
a_{j_21}^i & a_{j_22}^i & \cdots & a_{j_2i_k}^i \\
\vdots & \vdots & \ddots & \vdots \\
a_{j_{i_k}1}^i & a_{j_{i_k}2}^i & \cdots & a_{j_{i_k}i_k}^i
\end{pmatrix} . \tag{12}
\]

Thus, the \( i \)-th block diagonal matrix \( D_i^g \) of \( C^g \) can be solved as

\[
D_i^g = \begin{pmatrix}
b_{j_11}^i & b_{j_12}^i & \cdots & b_{j_1i_k}^i \\
b_{j_21}^i & b_{j_22}^i & \cdots & b_{j_2i_k}^i \\
\vdots & \vdots & \ddots & \vdots \\
b_{j_{i_k}1}^i & b_{j_{i_k}2}^i & \cdots & b_{j_{i_k}i_k}^i
\end{pmatrix}^{-1} . \tag{12}
\]

After all of the block diagonal matrices are computed, we can get the representation matrix \( C^g \) of the symmetry \( g \). \( \square \)

The above theorem can also be regarded as a consequence of the commutativity constraint between the functional representation and the Laplace-Beltrami operator [OBCS’12], which means that \( C^g \) must be block diagonal and can be recoverable from \( i_k \) linearly independent constraints on each block. Theorem 3.1 proposes a new band-by-band method to compute the functional representation using a much smaller number of constraints. In practice, we divide
Figure 4: Reflectional global intrinsic symmetry of a C_2 shape is recovered from four symmetric pairs of points shown in (a), where (b)-(d) are different symmetry representations, that is, point-to-point correspondence (b), action of transferring colors from left to right (c), and the representation matrix (d), respectively.

The condition of symmetry determination in the above corollary is not strict. Because the probability of choosing m points that are not full rank in the function space W_i, i.e., their spectral embedding vector restricted on the i_k (m ≥ i_k) dimension space W_i has rank lower than i_k is very low. This is similar to the probability of randomly choosing more than three 3D points which are exactly on a line is very low. If the m points are not full rank, a small turbulence will make them full rank. Furthermore, in addition to the above indicator functions, we can also add the global intrinsic symmetry invariant functions, such as Heat Kernel Signature and Wave Kernel Signature, to make the constraint functions full rank for the computation of symmetry representation matrix in practice.

Corollary 3.2. For a generic compact manifold M, the global intrinsic symmetry g can be uniquely determined from only one pair of points (p, g(p)) with the condition that p is a generic point. Furthermore, each block diagonal matrix D^i of C^i is one dimension, i.e., D^i = 1 or -1.

The above corollary coincides with a result for the Heat Kernel Map [OMMG10] from a different view, i.e., the global intrinsic symmetry on a generic manifold can be uniquely recovered from only one generic point and its symmetric point. It is also proven that the area of non-generic points is zero, i.e., the generic points are almost everywhere on the manifold [Che76, OMMG10]. However, their method can only recover the point-to-point correspondences of the symmetry, while our method can compute the more general functional representation.

4. Algorithm

In this section, based on the continuous theoretical analysis above, we define an algorithm for effectively computing group representations of global intrinsic symmetries of manifolds discretized as triangular meshes in R^2. If the global intrinsic symmetry group G(M) of shapes only have two elements, i.e., G(M) = {I, g|g^2 = I}, we denote them as C_2 shapes. In this paper, symmetric shapes are classified into two categories, C_2 shapes and others with more symmetries. There is no need to classify symmetric pairs of points in the first situation, where is only one reflection beside the identity symmetry, e.g., human and four-legged animals. As far as we know, most previous works can only handle shapes in this class.
4.1. C_2 shapes

The proposed algorithm consists primarily of the following steps: pairs of points generation, computation of representation matrix and refinement by Iterative Closest Point (ICP) algorithm.

**Step 1: pairs of points generation and functional constraints.** To compute the only non-trivial reflectional symmetry \( g \) in the \( C_2 \) shapes, some symmetric pairs of points \((p, g(p))\) should be prepared firstly, where \( g(p) \) is also a symmetric pair for \( g \). Based on Theorem 3.1 and Corollary 3.1, the number of symmetric pairs of points should be larger than the bounded maximum multiplicity \( m \) of the eigenvalues of the Laplace-Beltrami operator.

In practice, we find that \( m \) is seldom larger than four for the first smallest eigenvalues that are used and generate two mutual symmetric pairs coherently, i.e., \((p_i, g(p_i)), (g(p_i), p_i), i = 1, 2, \) which are selected as the two closest pairs from the local extrema of the Heat Kernel Signature (HKS) with a larger time \( t = 4/n^2 \) defined in [SOG09]. The closeness of HKS is measured by their Euclidean distances. To improve the stability, we don’t select the closest pairs, i.e., the geodesic distances between points \( p_1, g(p_1), p_2, \) and \( g(p_2) \) should be enough. The selected symmetric pairs of points are usually at the tips of the hands or legs of a human or animal as illustrated in Figure 4(a), which are enough to recover the reflectional symmetry with different representations shown in Figures 4(b)-(d). In practice, the indicator functions of small geodesic disks of the above four points, \( p_1, p_2, g(p_1), \) and \( g(p_2) \), are used as functional constraints for computing a functional map. In addition to the four indicator functions above, one hundred scaled HKSs and WKSs are also added as functional constraints.

**Step 2: computation of the representation matrix.** Unlike previous works that compute the functional map matrix entirely, we compute the symmetry representation matrix \( C \) band-by-band. In practice, the coefficient vectors of the constrained functions in the space \( W \) are not exactly aligned with the orthogonal matrix \( D^f \) in Equation (11). The best orthogonal matrix \( D^f \) is computed via the Singular Value Decomposition (SVD), i.e., \( D^f = U \cdot V^T \) where \([U, S, V] = SVD(B + A')\). The first four columns of matrices \( A \) and \( B \) are coefficient vectors of the four indicator functions and their symmetric ones respectively. The other columns of both \( A \) and \( B \) are coefficient vectors of 100 HKSs and WKSs with the scaled parameter 0.01 as constraints.

**Step 3: ICP refinement.** Since the computed matrix \( C \) can be regarded as a rigid alignment between the spectral embeddings of all points on shapes as defined in Equation (8), we are able to refine it by the iterative closest point algorithm as in Section 6.2 of the original functional map [OBCS12]. During the refinement process, the point-to-point correspondence of the global intrinsic symmetry is obtained at the same time.

4.2. Shapes with more symmetries

Because global intrinsic symmetries are represented as matrices and can be recovered from sparse pairs of points, we vote and cluster symmetries in the matrix space like extrinsic symmetry extraction [MGPO6]. Therefore, our algorithm pipeline consists of points sampling, pairs of points generation and pruning, and clustering in transformation space, as illustrated in Figure 5.

**Step1: symmetric points sampling.** Although all pairs of points on shapes \( M \) can be considered, in practice, we use all possible pairs of points of a smaller symmetry-invariant set \( P = \{p_i \in M, i = 1, 2, \ldots, n\} \) satisfying \( g(P) \) for \( g \in G(M) \) [KLC10]. The set \( \{g(p), \forall g \in G(M)\} \) from a given point \( p \) is called a symmetry orbit [LCDF10], a specific type of symmetry-invariant set. Furthermore, a symmetry-invariant set can be seen as a union of some symmetry orbits. Some recent works [KLC10, LCDF10, WSSZ14] have conducted extracting symmetry-invariant sets or orbits.

The number of sampling points \( n \) should be larger than the maximum multiplicity \( m \) of the eigenvalues of the Laplace-Beltrami operator. In practice, we find that the number of points in the symmetry orbit of a point other than the stationary point under all symmetries, is usually larger than the bounded maximum multiplicity \( m \). In this paper, the symmetry orbit from a single point is used as symmetric point sampling. We use the method of [WSSZ14] to compute the symmetry orbit. The single point for symmetry orbit computation can be random selected or defined by user.

**Step 2: pairs of points generation and matrix computation.** If the symmetry-invariant set has \( n \) points, then there are \( C(n, m) = C(n, m) \) possible \( m \) pairs of points \((p, q_i), i = 1, 2, \cdots, m\). Since both \( n \) and \( m \) are small in practice, it is possible to check all possibilities. However, several pairs can be pruned out based on geodesic distance preservation. If one of the ratios \( \frac{d(p, q_i)}{\|p - q_i\|} \) (or the reciprocal) is smaller than a threshold \( \delta \) for \( i, j = 1, 2, \cdots, m \) and...
In this paper, we use the classical cotangent weight scheme \cite{MDS02} without area normalization for the discretization of the Laplace-Beltrami operator, which is less sensitive to volume distortion and provides a more compact representation matrix, which is also refined by the ICP algorithm.

Implementation details. In this paper, we use the classical cotangent weight scheme \cite{MDS02} without area normalization for the discretization of the Laplace-Beltrami operator, which is less sensitive to volume distortion and provides a more compact representation matrix, which is also refined by the ICP algorithm.

5. Numerical results and applications

Table 1: Evaluation results on TOSCA data sets based on the manually selected ground-truth set used in \cite{KLC10}.

<table>
<thead>
<tr>
<th></th>
<th>MT</th>
<th>BIM</th>
<th>OFM</th>
<th>Our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cat</td>
<td>66</td>
<td>93.7</td>
<td>90.9</td>
<td>96.5</td>
</tr>
<tr>
<td>Centaur</td>
<td>92</td>
<td>100</td>
<td>96.0</td>
<td>92.0</td>
</tr>
<tr>
<td>David</td>
<td>82</td>
<td>97.4</td>
<td>94.8</td>
<td>92.5</td>
</tr>
<tr>
<td>Dog</td>
<td>91</td>
<td>100</td>
<td>93.2</td>
<td>97.4</td>
</tr>
<tr>
<td>Horse</td>
<td>92</td>
<td>97.1</td>
<td>95.2</td>
<td>99.4</td>
</tr>
<tr>
<td>Michael</td>
<td>87</td>
<td>98.9</td>
<td>94.6</td>
<td>91.4</td>
</tr>
<tr>
<td>Victoria</td>
<td>83</td>
<td>98.3</td>
<td>98.7</td>
<td>95.5</td>
</tr>
<tr>
<td>Wolf</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Gorilla</td>
<td>-</td>
<td>98.9</td>
<td>98.9</td>
<td>100</td>
</tr>
<tr>
<td>Average</td>
<td>85</td>
<td>98.0</td>
<td>95.1</td>
<td>94.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>MT</th>
<th>BIM</th>
<th>OFM</th>
<th>Our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr rate (%)</td>
<td>62</td>
<td>84.8</td>
<td>91.7</td>
<td>94.5</td>
</tr>
<tr>
<td>Mesh rate (%)</td>
<td>71.8</td>
<td>76.1</td>
<td>97.2</td>
<td>98.6</td>
</tr>
</tbody>
</table>

Table 2: Evaluation results on SCAPE data sets based on the manually selected ground-truth set used in \cite{KLC10}.

We set $\epsilon = \sqrt{\frac{\text{area}(M)}{20K}}$ and $\beta = 75\%$ by default, which are the same as the previous works \cite{KLC10,LLL15}.

There are benchmarks of reflectional symmetries \cite{KLC10} available on both TOSCA \cite{BBK08} and SCAPE \cite{ASK05} data sets, where ground truth symmetric points of $T$ are manually given for each shape. We compare the proposed global intrinsic symmetry detection method for $C_2$ shapes on the two data sets with the state-of-the-art previous works, Möbius Transformations (MT) \cite{KLC10}, Blended Intrinsic Maps (BIM) \cite{KFL11}, and Orthonormal Functional Maps (OFM) \cite{LLL15}, in Table 1 and Table 2 respectively using the above two metrics. Our method clearly obtains the best performance on the SCAPE data set and comparable results on the TOSCA data set with much less computational effort. It is because our band-by-band computation has only a small number of constraints and is not required to solve complex optimization problems. Our MATLAB implementation takes 24 minutes to compute the global symmetries for all meshes on the TOSCA data set, while the times of OFM and BIM are more than one hour and six hours as indicated in \cite{LLL15}. Some point-to-point correspondences of our detected symmetries on the SCAPE and TOSCA data sets are illustrated in Figures 6 and 7, respectively.

Evaluation. All of the previous works mainly detect reflectional symmetries, so there are no benchmarks available for evaluating rotational global intrinsic symmetries. To evaluate the performance of our method on symmetric shapes with symmetry groups other than $C_2$, we use the five-pointed star model in Figure 8 for an example, where the known twenty extrinsic symmetries serve as ground truth. One hundred points $T = \{ p_i | i = 1, 2, \ldots, 100 \}$ obtained by furthest point sampling are used for evaluation. For each global intrinsic symmetry $g$, we compute the mean geodesic distance errors between our computed intrinsic symmetry point $g(p_i)$ and its extrinsic ground-truth $G_C(p_i), i = 1, 2, \ldots, 100$, which are recorded under each symmetry shown in Figure 8. It can be seen that the mean errors of the twenty symmetries are far smaller than the tolerance error $\epsilon = 3.92$, and all of the mesh rates are 100%.
Figure 6: Some symmetry detection results of our method on the SCAPE data sets [ASK*05].

Figure 7: Some symmetry detection results of our method on the TOSCA data sets [BBK08].

Figure 8: Mean geodesic distance errors of the computed twenty global intrinsic symmetries (under each symmetry), which are much smaller than the tolerance error $\varepsilon = 3.92$. The left column is the original color both in front and back view, while the other figures demonstrate symmetries by transferring the original color onto the shape itself in front view.

Figure 9: The computed eight global intrinsic symmetries on a Table model, where each symmetry is demonstrated by transferring the original color (middle) onto the shape itself. The composition operation table of this global intrinsic symmetry group is show in Table 3.
Our proposed method performs comparably or slightly better than

### Table 3: Group operation table of the global intrinsic symmetry group of the Table model shown in Figure 9, where the element in the i-th row and j-th column is the composition symmetry $g_j \cdot g_i$.

<table>
<thead>
<tr>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
<th>$g_5$</th>
<th>$g_6$</th>
<th>$g_7$</th>
<th>$g_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_1$</td>
<td>$g_2$</td>
<td>$g_3$</td>
<td>$g_4$</td>
<td>$g_5$</td>
<td>$g_6$</td>
<td>$g_7$</td>
<td>$g_8$</td>
</tr>
<tr>
<td>$g_2$</td>
<td>$g_3$</td>
<td>$g_4$</td>
<td>$g_5$</td>
<td>$g_6$</td>
<td>$g_7$</td>
<td>$g_8$</td>
<td>$g_1$</td>
</tr>
<tr>
<td>$g_3$</td>
<td>$g_4$</td>
<td>$g_5$</td>
<td>$g_6$</td>
<td>$g_7$</td>
<td>$g_8$</td>
<td>$g_1$</td>
<td>$g_2$</td>
</tr>
<tr>
<td>$g_4$</td>
<td>$g_5$</td>
<td>$g_6$</td>
<td>$g_7$</td>
<td>$g_8$</td>
<td>$g_1$</td>
<td>$g_2$</td>
<td>$g_3$</td>
</tr>
<tr>
<td>$g_5$</td>
<td>$g_6$</td>
<td>$g_7$</td>
<td>$g_8$</td>
<td>$g_1$</td>
<td>$g_2$</td>
<td>$g_3$</td>
<td>$g_4$</td>
</tr>
<tr>
<td>$g_6$</td>
<td>$g_7$</td>
<td>$g_8$</td>
<td>$g_1$</td>
<td>$g_2$</td>
<td>$g_3$</td>
<td>$g_4$</td>
<td>$g_5$</td>
</tr>
<tr>
<td>$g_7$</td>
<td>$g_8$</td>
<td>$g_1$</td>
<td>$g_2$</td>
<td>$g_3$</td>
<td>$g_4$</td>
<td>$g_5$</td>
<td>$g_6$</td>
</tr>
<tr>
<td>$g_8$</td>
<td>$g_1$</td>
<td>$g_2$</td>
<td>$g_3$</td>
<td>$g_4$</td>
<td>$g_5$</td>
<td>$g_6$</td>
<td>$g_7$</td>
</tr>
</tbody>
</table>

### Table 4: The corresponding inverse of the group operation table shown in Table 3.

<table>
<thead>
<tr>
<th>Inverse</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
<th>$g_5$</th>
<th>$g_6$</th>
<th>$g_7$</th>
<th>$g_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

### Characterization of symmetry group.
Our method can detect all reflectional and rotational global intrinsic symmetries of a 3D shape and represent them as matrices. Since the composition operation of symmetries now becomes a much simpler matrix product operation of their representation matrices, we are able to efficiently apply linear algebra methods to characterize the symmetry group.

When all of the symmetry representation matrices are computed, we can easily obtain the symmetry group table, which describes the structure of the finite symmetry group by arranging all of the possible composition results of the symmetries. In practice, the infinite symmetry representation matrix is approximated by a finite dimensional orthogonal matrix used in the clustering might be replaced with possibly more robust measures, for example, the logarithm mapping for describing 3D extrinsic symmetry [SAD+16].

### Acknowledgments
We thank the anonymous reviewers for their constructive comments. We also thank Maks Ovsjanikov for sharing the Ocotopus models used in Figures 1 and 3, and Alexander Bronstein et al. for sharing the codes of computing geodesic distances. This work was supported in part by NSFC (61402300, 61373160, 61522213, 61379090), 973 Program (2015CB352501), Guangdong Science and Technology Program (2014TX01X033, 2015A030312015, 2016A050503036), Shenzhen Innovation Program (JCYJ20151015151249564, 827-000196) and Excellent Young Scholar Fund of Shijiazhuang Tiedao University.

### References


© 2017 The Author(s)

Computer Graphics Forum © 2017 The Eurographics Association and John Wiley & Sons Ltd.


